

On dually almost MRD codes

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Abstract

In this paper we define and study a family of codes which come close to be MRD codes, so we call them AMRD codes (almost MRD). An AMRD code is a code with rank defect equal to 1. AMRD codes whose duals are AMRD are called dually AMRD. Dually AMRD codes are the closest to the MRD codes given that both they and their dual codes are almost optimal. Necessary and sufficient conditions for the codes to be dually AMRD are given. Furthermore we show that dually AMRD codes and codes of rank defect one and maximum 2-generalized weight coincide when the size of the matrix divides the dimension.

1 Introduction

Rank metric codes have cryptographic applications and applications in tape recording. Recently it was shown how to employ them for error correction in coherent linear network coding ([9], [16] [17]). Due to these applications, there is a steady stream of work that focuses on general properties of codes with rank metric.

There exist two representations of rank metric codes: matrix representation and vector representation. In matrix representation linear rank metric codes are \mathbb{F}_q -linear subspaces of $(\mathbb{F}_q)_{n,m}$, where the norm of an element $A \in (\mathbb{F}_q)_{n,m}$ is defined as the rank of the matrix. In vector representation, rank metric codes are \mathbb{F}_{q^m} -linear subspaces of the vector space $\mathbb{F}_{q^m}^n$, where the norm of a vector $v \in \mathbb{F}_{q^m}^n$ is defined as the maximal number of coordinates of v which are linearly independent over \mathbb{F}_q .

An MRD code is a rank metric code which is maximal in size given the minimum distance, or in other words it achieves the Singleton bound for the rank metric distance. Delsarte [3] and independently Gabidulin [8] proved the existence of \mathbb{F}_q -linear MRD codes

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for all q, m, n and dimension $1 \leq t \leq mn$ divisible by m . Given the parameters q, m, n, k , the code $C \leq \mathbb{F}_{q^m}^n$, they describe has a particular construction through a generator matrix $M_k(v)$ and is called in the literature the Gabidulin code \mathcal{G} . Recently new constructions of MRD codes have been found which are not equivalent to Gabidulin codes \mathcal{G} ([2, 15]).

In analogy with the Singleton defect for classical codes, in [1] the authors propose a definition of rank defect for \mathbb{F}_q -linear rank metric codes. The rank defect of a code $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ measures how far \mathcal{C} is away from being a MRD code. Based on this concept a QMRD code is defined in [1] as an \mathbb{F}_q -linear code $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ with rank defect 0 and which is not MRD, i.e \mathcal{C} has rank defect 0 and $m \nmid t$.

In this work we define and study a family of codes which come close to be MRD codes, called dually almost MRD codes or simply dually AMRD codes. This paper is structured as follows. In Section 2 we give the preliminaries on rank metric codes, rank defect, QMRD codes, rank distribution and generalized weights. In Section 3 we present the definition of dually AMRD codes, we give necessary and sufficient conditions for its existence based on the parameters n, m, t and d , moreover we establish its existence for the case $m = t$.

Using the rank distribution in Section 4 we give sufficient and necessary conditions for the code to be dually AMRD. In particular we establish a relationships between the number of vectors of minimal weight of the original code \mathcal{C} and its dual code \mathcal{C}^\perp , which guarantees that the code is dually AMRD. We also analyze the self-dual AMRD codes.

Finally, in Section 5 we study the generalized weights for \mathbb{F}_q -linear codes. In this part we establish relations between the generalized weights of an \mathbb{F}_q -linear code and its rank defect. We also prove that, when m divides the dimension t , the concept of dually AMRD for \mathbb{F}_q -linear codes coincides with the concept of 2-AMRD for \mathbb{F}_q -linear codes (or near almost codes).

2 Preliminaries

Let \mathbb{F}_q denote a finite field with q elements and let $V = (\mathbb{F}_q)_{n,m}$ be the \mathbb{F}_q -vector space of matrices over \mathbb{F}_q of type (n, m) . On V we define the so-called rank metric distance by $d(A, B) = \text{rank}(A - B)$ for $A, B \in V$.

A t -dimensional \mathbb{F}_q -subspace $\mathcal{C} \leq V$ endowed with the metric d is called a \mathbb{F}_q -linear rank metric code with minimum distance $d(\mathcal{C}) = \min \{d(A, B) \mid A \neq B \in \mathcal{C}\}$. Clearly, the minimum distance of a code $\mathcal{C} \neq \{0\}$ is also

$$d(\mathcal{C}) := \min\{\text{rank}(A) : A \in \mathcal{C}, A \neq 0\}.$$

Similarly, as in the classical coding theory, the rank distribution of \mathcal{C} is the collection $(A_i(\mathcal{C}))_{i \in \mathbb{N}}$, where $A_i(\mathcal{C}) := |\{A \in \mathcal{C} : \text{rank}(A) = i\}|$ for $i \in \mathbb{N}$. The dual of a code $\mathcal{C} \leq V$ is the code

$$\mathcal{C}^\perp := \{B \in (\mathbb{F}_q)_{n,m} : \text{Tr}(BA^t) = 0 \text{ for all } A \in \mathcal{C}\}.$$

A code $\mathcal{C} \leq V$ is *self-dual* provided $\mathcal{C} = \mathcal{C}^\perp$. As $\dim_{\mathbb{F}_q}(\mathcal{C}) + \dim_{\mathbb{F}_q}(\mathcal{C}^\perp) = \dim_{\mathbb{F}_q} V = nm$ a self-dual code \mathcal{C} has dimension $\frac{nm}{2}$.

The field \mathbb{F}_{q^m} may be viewed as an m -dimensional vector space over \mathbb{F}_q . The *rank* of a vector $v = (v_1, \dots, v_n) \in \mathbb{F}_{q^m}^n$ is defined as the maximum number of coordinates in v that

are linearly independent over \mathbb{F}_q , i.e. $\text{rank}(v) := \dim_{\mathbb{F}_q} \langle v_1, \dots, v_n \rangle$. Then we have a rank metric distance given by $d(v, u) = \text{rank}(v - u)$ for $v, u \in \mathbb{F}_{q^m}^n$. An \mathbb{F}_{q^m} -linear subspace $C \leq \mathbb{F}_{q^m}^n$ of dimension k endowed with this metric is called an \mathbb{F}_{q^m} -linear rank-metric $[n, k]$ code. The minimum distance of a code $C \neq \{0\}$ is

$$d(C) := \min\{\text{rank}(v) : v \in C, v \neq 0\}.$$

A code $C \leq \mathbb{F}_{q^m}^n$ is *self-dual* provided $C = C^\perp$, where C^\perp is defined with respect to the standard inner product of $\mathbb{F}_{q^m}^n$.

Let $v = (v_1, \dots, v_n) \in \mathbb{F}_{q^m}^n$, and let $\mathcal{B} = \{\gamma_1, \dots, \gamma_m\}$ be a basis of \mathbb{F}_{q^m} over \mathbb{F}_q . The \mathbb{F}_q -linear code associated to an \mathbb{F}_{q^m} -linear code $C \leq \mathbb{F}_{q^m}^n$ with respect to the basis \mathcal{B} is

$$\lambda_{\mathcal{B}}(C) := \{\lambda(v) : v \in C\},$$

where $\lambda(v) = (\lambda_{i,j}) \in (\mathbb{F}_q)_{n,m}$ is the matrix such that $v_i = \sum_{j=1}^m \lambda_{i,j} \gamma_j$ for all $i = 1, \dots, n$.

It is well known that the rank distributions of C and $\lambda_{\mathcal{B}}(C)$ agree and $\dim_{\mathbb{F}_q}(\lambda_{\mathcal{B}}(C)) = m \cdot \dim_{\mathbb{F}_{q^m}}(C)$. In general $C^\perp \neq \lambda_{\mathcal{B}}(C)^\perp$, however it has been shown in [13] that their rank distributions also agree.

Theorem 2.1 ([3], Theorem 5.6). (SINGLETON-BOUND) *Let $C \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code of dimension t with minimum distance d . Then we have*

$$d \leq \min\{n - t/m + 1, m - t/m + 1\}.$$

In particular if $C \leq \mathbb{F}_{q^m}^n$ is an \mathbb{F}_{q^m} -linear code of dimension k , then

$$d \leq \min\{n - k + 1, \frac{m}{n}(n - k) + 1\}.$$

Rank metric codes meeting the Singleton bound are called *Maximum Rank Distance* (MRD) codes. Delsarte was the first who proved in [3] the existence of linear MRD codes.

Given a vector $v = (v_1, \dots, v_n) \in \mathbb{F}_{q^m}^n$ we denote by $M_k(v) \in (\mathbb{F}_{q^m})_{k,n}$ the matrix

$$M_k(v) = \begin{pmatrix} v_1 & v_2 & \dots & v_n \\ v_1^{[1]} & v_2^{[1]} & \dots & v_n^{[1]} \\ & & \vdots & \\ v_1^{[k-1]} & v_2^{[k-1]} & \dots & v_n^{[k-1]} \end{pmatrix},$$

where $[i] := q^i$.

Gabidulin showed in [8] that if v_1, \dots, v_n are linearly independent over \mathbb{F}_q , then the \mathbb{F}_{q^m} -linear code $C \leq \mathbb{F}_{q^m}^n$ generated by the matrix $M_k(v_1, \dots, v_n)$ is a k -dimensional MRD code and we call it the *Gabidulin code \mathcal{G} generated by $M_k(v_1, \dots, v_n)$* .

Remark 2.2. Throughout the paper, d and d^\perp denote the minimum distance of the code and its dual respectively. Furthermore, in this work we assume $n \leq m$, therefore $d \leq n - t/m + 1$ for an \mathbb{F}_q -linear code of dimension t and $d \leq n - k + 1$ for an \mathbb{F}_{q^m} -linear code of dimension k . Also, unless stated otherwise, we will only consider non-trivial codes, i.e. $\{0\} \neq C \neq (\mathbb{F}_q)_{n,m}$ and $\{0\} \neq C \neq \mathbb{F}_{q^m}^n$.

In analogy with the Singleton defect for classical codes given in [7], we have the following definition for the rank defect of rank metric codes.

Definition 2.3 ([1]). The *rank defect* of an \mathbb{F}_q -linear code $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ is defined by

$$\text{Rdef}(\mathcal{C}) = n - \left\lceil \frac{t}{m} \right\rceil + 1 - d.$$

If $C \leq \mathbb{F}_{q^m}^n$ is an \mathbb{F}_{q^m} -linear $[n, k, d]$ code, then the rank defect of C is defined as the defect of the associated code $\lambda(C)$, i.e. $\text{Rdef}(C) = n - k + 1 - d$.

Note that $\text{Rdef}(\mathcal{C}) = 0$ if \mathcal{C} is MRD. However, $\text{Rdef}(\mathcal{C})$ may be zero also for codes \mathcal{C} which are not MRD. These codes are the closest codes to the MRD codes and are called quasi MRD codes (see [1]). Concretely, we have the following definition.

Definition 2.4 ([1]). A code \mathcal{C} of dimension t is *Quasi-MRD*, or *QMRD*, if $m \nmid t$ and $\text{Rdef}(\mathcal{C}) = 0$.

In [1] it is shown that as for MRD codes QMRD codes exist for all choices of the parameters $1 \leq n \leq m$ and $1 \leq t < nm$ such that $m \nmid t$.

Generalized weights for \mathbb{F}_{q^m} -linear codes were introduced in [10, 12]. It was proved in [5] that, through a refinement, the definition given in [12] agrees with the definition of [10]. Similarly as V.K. Wei, who in [18] studied the generalized weights for codes with Hamming metric motivated by cryptographical applications in the wire-tap channel of type II, the authors in [10, 12] introduce generalized rank weights to study the equivocation of wire-tap codes for network coding.

Definition 2.5 ([10]). Given an \mathbb{F}_{q^m} -linear code $C \leq \mathbb{F}_{q^m}^n$ of dimension k and an integer $1 \leq r \leq k$ the r^{th} -generalized weight of C is

$$\mathcal{M}_r(C) := \min\{\dim_{\mathbb{F}_{q^m}}(V) : V \in \Gamma(\mathbb{F}_{q^m}^n), \dim_{\mathbb{F}_{q^m}}(V \cap C) \geq r\},$$

where $\Gamma(\mathbb{F}_{q^m}^n) := \{V \leq \mathbb{F}_{q^m}^n : V^q = V\}$ and $V^q := \{v^q := (v_1^q, \dots, v_n^q) : v \in V\}$.

The following theorem summarizes the main properties of the generalized weights for \mathbb{F}_{q^m} -linear codes, which are similar to the generalized weights for codes with the Hamming metric given by V.K. Wei in [18].

Theorem 2.6 ([5], [10]). Let $C \leq \mathbb{F}_{q^m}^n$ be an \mathbb{F}_{q^m} -linear code of dimension k . Then we have

1. $\mathcal{M}_1(C) = d(C)$.
2. $\mathcal{M}_k(C) \leq n$.
3. For any $1 \leq r \leq k$, we have $\mathcal{M}_r(C) < \mathcal{M}_{r+1}(C)$.
4. For every $1 \leq r \leq k$, we have $\mathcal{M}_r(C) \leq n - k + r$.

5. $\{\mathcal{M}_1(C), \dots, \mathcal{M}_k(C)\} = [n] \setminus \{n+1 - \mathcal{M}_{n-k}(C^\perp), n+1 - \mathcal{M}_1(C^\perp)\}$, where $[n] := \{1, \dots, n\}$.

For \mathbb{F}_q -linear codes the generalized weights were introduced in [14], refining previous definitions for \mathbb{F}_{q^m} -linear codes given in [10, 12, 5] and considering an anticode approach.

Definition 2.7 ([14]). An *optimal anticode* $\mathcal{A} \leq (\mathbb{F}_q)_{n,m}$ is an \mathbb{F}_q -linear code such that $\dim(\mathcal{A}) = m \cdot \maxrk(\mathcal{A})$, where $\maxrk(\mathcal{A}) := \max\{\text{rk}(M) : M \in \mathcal{A}\}$.

Given an \mathbb{F}_q -linear code \mathcal{C} of dimension t and an integer $1 \leq r \leq t$, the r -th *generalized weight* of \mathcal{C} is

$$a_r(\mathcal{C}) := \min\{\maxrk(\mathcal{A}) : \mathcal{A} \subseteq (\mathbb{F}_q)_{n,m} \text{ is an optimal anticode with } \dim(\mathcal{C} \cap \mathcal{A}) \geq r\}.$$

Theorem 2.8 ([14]). *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code of dimension t . The following hold:*

1. $a_1(\mathcal{C}) = d(\mathcal{C})$.
2. $a_t(\mathcal{C}) \leq n$.
3. For any $1 \leq r \leq t-1$, we have $a_r(\mathcal{C}) \leq a_{r+1}(\mathcal{C})$.
4. For any $1 \leq r \leq t-m$, we have $a_r(\mathcal{C}) < a_{r+m}(\mathcal{C})$.
5. For any $1 \leq r \leq t$, we have $a_r(\mathcal{C}) \leq n - \lfloor \frac{t-r}{m} \rfloor$.

The following theorem, which was proven in [14], shows that for \mathbb{F}_q -linear codes the generalized weights refine, as an algebraic invariant, generalized rank weights of \mathbb{F}_{q^m} -linear codes.

Theorem 2.9 ([14]). *Let $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ be an \mathbb{F}_{q^m} -linear code of dimension k . For any basis \mathcal{B} of \mathbb{F}_{q^m} over \mathbb{F}_q and for any integers $1 \leq r \leq t$ and $0 \leq \epsilon \leq m-1$, we have $m_r(\mathcal{C}) = a_{rm-\epsilon}(\lambda_{\mathcal{B}}(\mathcal{C}))$.*

Remark 2.10. We often write a_r or a_r^\perp to denote $a_r(\mathcal{C})$ or $a_r(\mathcal{C}^\perp)$ respectively.

Theorem 2.11 ([14]). *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code of dimension t and \mathcal{C}^\perp its dual code. Then we have*

$$\{n+1 - a_{1+t-m}(\mathcal{C}), \dots, n+1 - a_{1+t-\lfloor t/m \rfloor m}(\mathcal{C})\} = [n] \setminus \{a_1(\mathcal{C}^\perp), \dots, a_{1+(n-\lfloor \frac{t+1}{m} \rfloor)m}(\mathcal{C}^\perp)\},$$

where $[n] := \{1, \dots, n\}$.

3 Dually almost MRD codes

We want to study the class of codes which are close to being MRD. In analogy with the definition of almost MDS codes (see [7]), we define:

Definition 3.1. The code \mathcal{C} is an s -almost MRD code or $A^s\text{MRD}$ if and only if $\text{Rdef}(\mathcal{C}) = s$. $A^1\text{MRD}$ codes are simply called AMRD codes. Equivalently a code \mathcal{C} is an AMRD code if and only if $d = n - \lceil t/m \rceil$.

It is known that MRD and QMRD \mathbb{F}_q -linear codes exist for all m, n, t, q . Next we see that AMRD codes exist also for these parameters in the case $m \mid t$.

Lemma 3.2 (Existence of AMRD codes). *If $\mathcal{G} \leq \mathbb{F}_{q^m}^n$ is the Gabidulin code generated by $M_k(v_1, \dots, v_n)$, then the extended code $\widehat{\mathcal{G}}$ is an \mathbb{F}_{q^m} -linear AMRD code with minimum distance $\widehat{d} = n - k + 1$ and dual distance $\widehat{d}^\perp = 1$.*

Proof. One easily verifies that $d(C) = d(\widehat{C})$, for all \mathbb{F}_{q^m} -linear code $C \leq \mathbb{F}_{q^m}^n$. Therefore $n - k + 1 = d(\mathcal{G}) = d(\widehat{\mathcal{G}})$. Since $\dim_{\mathbb{F}_{q^m}} \widehat{\mathcal{G}} = k$, then $\text{Rdef}(\widehat{\mathcal{G}}) = (n + 1) - k + 1 - d(\widehat{\mathcal{G}}) = 1$. Furthermore, since $\bar{1} = (1, \dots, 1) \in \widehat{\mathcal{G}}^\perp$, we have $\widehat{d}^\perp = 1$. \square

Given an \mathbb{F}_q -linear AMRD code with $m \mid t$, the following lemma allows us to find \mathbb{F}_q -linear AMRD codes for which m does not divide the dimension t .

Lemma 3.3. *If $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ is an AMRD \mathbb{F}_q -linear code of dimension t with $m, t \neq 1$ and $m \mid t$, then there exists an AMRD \mathbb{F}_q -linear code $\mathcal{C}' \leq (\mathbb{F}_q)_{n,m}$ of dimension t' with $m \nmid t'$.*

Proof. If $m \mid t$, then it is always possible to find an integer t' such that $t' < t$ and $\lceil \frac{t'}{m} \rceil = \frac{t}{m}$. Since \mathcal{C} is AMRD, then $d(\mathcal{C}) = n - \frac{t}{m} = n - \lceil \frac{t'}{m} \rceil$. Let \mathcal{C}' be the t' -dimensional subcode of \mathcal{C} containing a vector of minimum rank d . Then \mathcal{C}' is AMRD. \square

It is well known that the dual code of an MRD code is also an MRD code and therefore both have rank defect 0. Based on this property we define.

Definition 3.4. We say that an \mathbb{F}_q -linear code \mathcal{C} is dually AMRD if $\text{Rdef}(\mathcal{C}) = \text{Rdef}(\mathcal{C}^\perp) = 1$. A similar definition is given for an \mathbb{F}_{q^m} -linear code $C \leq \mathbb{F}_{q^m}^n$ considering its associated code.

Example 3.5. Let $q = 2$, $n = m = 3$, $t = 4$ and

$$\mathcal{C} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

Then $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{C}^\perp$, $d = d^\perp = 1$ and \mathcal{C} is dually AMRD.

We can see that among all AMRD codes the dually AMRD codes are the most similar to MRD codes. However not all AMRD codes are dually AMRD, as we show in the following simple example.

Example 3.6. Let $q = 2$, $n = m = 2$, $t = 1$, $c = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ and

$$c^\perp = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then $d = 1$ and \mathcal{C} is AMRD, while $d^\perp = 1$ and \mathcal{C}^\perp is QMRD.

According to Proposition 19 in [1] an \mathbb{F}_{q^m} -linear code $C \leq \mathbb{F}_{q^m}^n$ is dually AMRD if and only if $d + d^\perp = n$. More generally we have the following results.

Proposition 3.7. *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code with minimum distance d and dual distance d^\perp . The following facts hold:*

1. *If \mathcal{C} is dually AMRD, then $t \geq m$ and $d + d^\perp = \begin{cases} n, & \text{if } m \mid t; \\ n - 1, & \text{if } m \nmid t. \end{cases}$*
2. *If $m \mid t$, then \mathcal{C} is dually AMRD if and only if $d + d^\perp = n$. In particular an \mathbb{F}_{q^m} -linear code C is dually AMRD if and only if $d + d^\perp = n$.*
3. *If $d + d^\perp = n$ and $m \nmid t$ or $d + d^\perp = n - 1$ and $m \mid t$, then \mathcal{C} is not a dually AMRD code.*
4. *Let $t = \beta m + \alpha$, where $\beta, \alpha \in \mathbb{Z}$ and $0 \leq \alpha < m$. Then we have.*
 - (a) *If $m \mid t$, then \mathcal{C} is dually AMRD if and only if $d = n - \beta$ and $d^\perp = \beta$.*
 - (b) *If $m \nmid t$, then \mathcal{C} is dually AMRD if and only if $d = n - \beta - 1$ and $d^\perp = \beta$.*
5. *Let $t = \beta m + \alpha$, where $\beta, \alpha \in \mathbb{Z}$ and $0 \leq \alpha < m$. Then \mathcal{C}^\perp is AMRD if and only if $d^\perp = \beta$.*

Proof. 1. Let \mathcal{C} be dually AMRD code. If $t < m$, then $\text{Rdef}(\mathcal{C}^\perp) = 0$, a contradiction. On the other hand,

$$1 = \text{Rdef}(\mathcal{C}) = n - \lceil t/m \rceil + 1 - d = \text{Rdef}(\mathcal{C}^\perp) = n - \lceil n - t/m \rceil + 1 - d^\perp = \lfloor t/m \rfloor + 1 - d^\perp.$$

$$\text{Therefore } d + d^\perp = n + \lfloor -t/m \rfloor + \lfloor t/m \rfloor = \begin{cases} n, & \text{if } m \mid t; \\ n - 1, & \text{if } m \nmid t. \end{cases}$$

2. Let $d + d^\perp = n$. Then

$$\begin{aligned} \text{Rdef}(\mathcal{C}) + \text{Rdef}(\mathcal{C}^\perp) &= (n - \lceil t/m \rceil + 1 - d) + (\lfloor t/m \rfloor + 1 - d^\perp) \\ &= \lfloor t/m \rfloor + \lfloor -t/m \rfloor + 2 \\ &= \begin{cases} 2, & \text{if } m \mid t; \\ 1, & \text{if } m \nmid t. \end{cases} \end{aligned}$$

We know that if $m \mid t$ and $\text{Rdef}(\mathcal{C}) = 0$, then \mathcal{C} is MRD and $\text{Rdef}(\mathcal{C}^\perp) = 0$, a contradiction. Therefore $\text{Rdef}(\mathcal{C}) = \text{Rdef}(\mathcal{C}^\perp) = 1$.

3. It is an immediate consequence of the part 1.

4. (a) Let $m \mid t$ and \mathcal{C} be dually AMRD. Then $d = n - \lceil t/m \rceil = n - \beta$ and $d^\perp = \lfloor t/m \rfloor = \beta$. The reciprocal is followed from part 2.
- (b) Let $m \nmid t$ and \mathcal{C} be dually AMRD. Then $d = n - \lceil t/m \rceil = n - (\beta + 1)$ and $d^\perp = \lfloor t/m \rfloor = \beta$. Reciprocally, let $d = n - \beta - 1$ and $d^\perp = \beta$. Since $t/m = \beta + \alpha/m$ with $0 < \alpha/m < 1$, then $\text{Rdef}(\mathcal{C}) = n - \lceil t/m \rceil + 1 - d = n - (\beta + 1) + 1 - n + \beta + 1 = 1$ and $\text{Rdef}(\mathcal{C}^\perp) = \lfloor t/m \rfloor + 1 - d^\perp = \beta + 1 - \beta = 1$.
5. By [1, Lemma 21] we have $\text{Rdef}(\mathcal{C}^\perp) = \beta + 1 - d^\perp$. Therefore the result easily follows. (Notice that $d^\perp = \beta$, implies $\beta \neq 0$, which is equivalent to $t > m$).

□

Remark 3.8. The reciprocal of the Proposition 3.7 (1) is not true when $m \nmid t$. In fact, if $d + d^\perp = n - 1$ and $m \nmid t$, then $\text{Rdef}(\mathcal{C}) + \text{Rdef}(\mathcal{C}^\perp) = 2$. Therefore $(\text{Rdef}(\mathcal{C}), \text{Rdef}(\mathcal{C}^\perp)) \in \{(0, 2), (2, 0), (1, 1)\}$ and \mathcal{C} is not necessarily a dually AMRD code. For example if $q = 2$, $m = n = 3$, $t = 7$ and

$$\mathcal{C}^\perp = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle,$$

then $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{C}$, $d = d^\perp = 1$ and \mathcal{C} is QMRD. This happens because $d \neq n - \beta - 1$ and $d^\perp \neq \beta$.

Remark 3.9. By Proposition 3.7 (2) an \mathbb{F}_{q^m} -linear code C and its associated code $\lambda_{\mathcal{B}}(C)$ are dually AMRD if and only if $d + d^\perp = n$. However not always a dually AMRD \mathbb{F}_q -linear code \mathcal{C} arises from a dually AMRD \mathbb{F}_{q^m} -linear code C , even when $m \mid t$ and $d + d^\perp = n$. For example the code $\mathcal{C} \leq (\mathbb{F}_2)_{3,3}$ with dimension $t = 3$, $d + d^\perp = 2 + 1 = 3$ and whose nonzero codewords are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

is a dually AMRD \mathbb{F}_2 -linear code which does not arise from an \mathbb{F}_{2^3} -linear code C . In fact, suppose $\lambda_{\mathcal{B}}(C) = \mathcal{C}$ for some code $C \leq \mathbb{F}_{2^3}^3$ and $\mathcal{B} = \{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of \mathbb{F}_{2^3}

over \mathbb{F}_2 . Then there exists a codeword $c \in C$ such that $c' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \lambda_{\mathcal{B}}(c)$.

Therefore $c = (\gamma_1, \gamma_2, 0) \in C$ and $0 \neq \gamma_1^{-1}c = (1, \gamma_1^{-1}\gamma_2, 0) \in C$. Then $0, c' \neq \lambda_{\mathcal{B}}(\gamma_1^{-1}c) = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{C}$, which is not possible.

Theorem 3.10 (Existence of 1-dimensional dually AMRD codes). *Dually AMRD \mathbb{F}_{q^m} -linear codes $C \leq \mathbb{F}_{q^m}^n$ with dimension $k = 1$ exist for all parameters m, n, q .*

Proof. Let v_1, \dots, v_{n-1} be linearly independent and \mathcal{G} the Gabidulin code generated by $M_1(v_1, \dots, v_{n-1})$. Then $\widehat{\mathcal{G}}$ is a 1-dimensional dually AMRD code. In fact, $d(\widehat{\mathcal{G}}) = d(\mathcal{G}) = n - 1$ and $d(\widehat{\mathcal{G}}^\perp) = 1$. Therefore $d(\widehat{\mathcal{G}}) + d(\widehat{\mathcal{G}}^\perp) = n$ and by Proposition 3.7 $\widehat{\mathcal{G}}$ is dually AMRD. \square

4 Rank distribution of dually AMRD \mathbb{F}_q -linear codes

In the following theorem the authors proved in [1] that the rank distribution of a code \mathcal{C} is determined by its parameters, together with the number of codewords of small weight: $A_d(\mathcal{C}), \dots, A_{n-d^\perp}(\mathcal{C})$.

Theorem 4.1 ([1]). *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be a t -dimensional code, with minimum distance d and dual minimum distance d^\perp . Let $\delta = 1$ if \mathcal{C} is MRD, and $\delta = 0$ otherwise. For all $1 \leq r \leq d^\perp$ we have*

$$\begin{aligned} A_{n-d^\perp+r}(\mathcal{C}) &= (-1)^r q^{\binom{r}{2}} \sum_{j=d^\perp}^{n-d} \begin{bmatrix} j \\ d^\perp - r \end{bmatrix} \begin{bmatrix} j - d^\perp + r - 1 \\ r - 1 \end{bmatrix} A_{n-j}(\mathcal{C}) \\ &\quad + \begin{bmatrix} n \\ d^\perp - r \end{bmatrix} \sum_{i=0}^{r-1-\delta} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n - d^\perp + r \\ i \end{bmatrix} \left(q^{t-m(d^\perp-r+i)} - 1 \right). \end{aligned}$$

In particular, n, m, t, d, d^\perp and $A_d(\mathcal{C}), \dots, A_{n-d^\perp}(\mathcal{C})$ determine the rank distribution of \mathcal{C} .

If we apply this theorem to dually AMRD \mathbb{F}_q -linear codes, then we have the following results.

Proposition 4.2. *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be a t -dimensional dually AMRD code, with minimum distance d and dual minimum distance d^\perp . The following facts hold:*

1. *If $m \nmid t$ and $t = \beta m + \alpha$ with $\beta, \alpha \in \mathbb{Z}$ and $1 \leq \alpha < m$, then*

$$\begin{aligned} A_{d+r}(\mathcal{C}) &= (-1)^{r-1} q^{\binom{r-1}{2}} \left(\begin{bmatrix} n-d-1 \\ r-1 \end{bmatrix} A_{d+1} + \begin{bmatrix} n-d \\ r \end{bmatrix} \begin{bmatrix} r-1 \\ 1 \end{bmatrix} A_d \right) \\ &\quad + \begin{bmatrix} n \\ d+r \end{bmatrix} \sum_{i=0}^{r-2} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} d+r \\ i \end{bmatrix} \left(q^{\alpha+m(r-i)} - 1 \right), \end{aligned}$$

for all $r = 2, \dots, n-d$.

2. *If $m \mid t$, then we have*

$$A_{d+r}(\mathcal{C}) = (-1)^r q^{\binom{r}{2}} \begin{bmatrix} n-d \\ r \end{bmatrix} A_d + \begin{bmatrix} n \\ d+r \end{bmatrix} \sum_{i=0}^{r-1} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} d+r \\ i \end{bmatrix} \left(q^{m(r-i)} - 1 \right),$$

for all $r = 1, \dots, n - d$. In particular, for an \mathbb{F}_{q^m} -linear code $C \leq \mathbb{F}_{q^m}^n$ of dimension k we have

$$A_{d+r}(C) = (-1)^r q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix} A_d + \begin{bmatrix} n \\ k-r \end{bmatrix} \sum_{i=0}^{r-1} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} d+r \\ i \end{bmatrix} (q^{m(r-i)} - 1),$$

for all $r = 1, \dots, n - d$.

3. If $m \mid t$, then $\frac{\begin{bmatrix} n \\ d+2 \end{bmatrix}}{q^{\begin{bmatrix} n-2 \\ 2 \end{bmatrix}}} (q^m - 1) \left(\begin{bmatrix} d+2 \\ 1 \end{bmatrix} - q^m - 1 \right) \leq A_d \leq \frac{\begin{bmatrix} n \\ d+1 \end{bmatrix}}{\begin{bmatrix} n-d \\ 1 \end{bmatrix}} (q^m - 1)$.

4. Let $m \mid t$. If $A_{d+1} = 0$, then $A_d = \frac{\begin{bmatrix} n \\ d+1 \end{bmatrix}}{\begin{bmatrix} n-d \\ 1 \end{bmatrix}} (q^m - 1)$ and if $A_{d+2} = 0$, then

$$A_d = \frac{\begin{bmatrix} n \\ d+2 \end{bmatrix}}{q^{\begin{bmatrix} n-2 \\ 2 \end{bmatrix}}} (q^m - 1) \left(\begin{bmatrix} d+2 \\ 1 \end{bmatrix} - q^m - 1 \right).$$

Proof. The proof of parts 1 and 2 are immediate by Theorem 4.1. The proof of parts 3 and 4 follow from

$$A_{d+1} = - \begin{bmatrix} n-d \\ 1 \end{bmatrix} A_d + \begin{bmatrix} n \\ d+1 \end{bmatrix} (q^m - 1) \geq 0$$

and

$$A_{d+2} = q^{\begin{bmatrix} n-d \\ 2 \end{bmatrix}} A_d + \begin{bmatrix} n \\ d+2 \end{bmatrix} (q^m - 1) \left(q^m + 1 - \begin{bmatrix} d+2 \\ 1 \end{bmatrix} \right) \geq 0.$$

□

In Theorem 35 [1] it was proved that if \mathcal{C} is dually AMRD and $m \mid t$, then the number of codewords of minimum rank in \mathcal{C} and \mathcal{C}^\perp are equal. We present a more general result through a different proof.

Lemma 4.3. *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be a t -dimensional dually AMRD code, with minimum distance d and dual minimum distance d^\perp . Then*

$$A_d = q^{t-m\lceil t/m \rceil} \left(\begin{bmatrix} n \\ \lceil t/m \rceil \end{bmatrix} + \begin{bmatrix} n - \lceil t/m \rceil + 1 \\ 1 \end{bmatrix} A_{\lceil t/m \rceil - 1}^\perp + A_{\lceil t/m \rceil}^\perp \right) - \begin{bmatrix} n \\ \lceil t/m \rceil \end{bmatrix}.$$

More precisely, we have:

1. $A_d = (q^{\alpha-m} - 1) \begin{bmatrix} n \\ \beta+1 \end{bmatrix} + q^{\alpha-m} \left(\begin{bmatrix} n-\beta \\ 1 \end{bmatrix} A_\beta^\perp + A_{\beta+1}^\perp \right)$ with $d^\perp = \beta$, if $m \nmid t$ and $t = m\beta + \alpha > m$, where $\beta, \alpha \in \mathbb{Z}$ and $0 \leq \alpha < m$.
2. $A_d = A_{d^\perp}^\perp$, if $m \mid t$.

Proof. We know that $d = n - \lceil t/m \rceil$ and $d^\perp = \lceil t/m \rceil$. By Theorem 31 [13] we have

$$\sum_{i=0}^{n-\nu} \begin{bmatrix} n-i \\ \nu \end{bmatrix} A_i = q^{t-m\nu} \sum_{j=0}^{\nu} \begin{bmatrix} n-j \\ \nu-j \end{bmatrix} A_j^\perp,$$

for all $0 \leq \nu \leq n$. In particular, for $\nu = \lceil t/m \rceil$ we have

$$\sum_{i=0}^{n-\lceil t/m \rceil} \begin{bmatrix} n-i \\ \lceil t/m \rceil \end{bmatrix} A_i = q^{t-m\lceil t/m \rceil} \sum_{j=0}^{\lceil t/m \rceil} \begin{bmatrix} n-j \\ \lceil t/m \rceil - j \end{bmatrix} A_j^\perp.$$

Therefore

$$\begin{bmatrix} n \\ \lceil t/m \rceil \end{bmatrix} + A_d = q^{t-m\lceil t/m \rceil} \left(\begin{bmatrix} n \\ \lceil t/m \rceil \end{bmatrix} + \begin{bmatrix} n - \lceil t/m \rceil + 1 \\ 1 \end{bmatrix} A_{\lceil t/m \rceil - 1}^\perp + A_{\lceil t/m \rceil}^\perp \right).$$

□

Theorem 4.4. *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be a t -dimensional AMRD code with $t = m\beta + \alpha > m$, where $\beta, \alpha \in \mathbb{Z}$ and $0 \leq \alpha < m$. The following hold:*

1. *If $m \nmid t$, then \mathcal{C} is dually AMRD if and only if*

$$A_d = (q^{\alpha-m} - 1) \begin{bmatrix} n \\ \beta + 1 \end{bmatrix} + q^{\alpha-m} \left(\begin{bmatrix} n - \beta \\ 1 \end{bmatrix} A_\beta^\perp + A_{\beta+1}^\perp \right) \quad \text{with } A_\beta^\perp \neq 0.$$

2. *If $m \mid t$, then \mathcal{C} is dually AMRD if and only if $A_d = A_{d^\perp}^\perp$.*

Proof. 1. Let $A_d = (q^{\alpha-m} - 1) \begin{bmatrix} n \\ \beta+1 \end{bmatrix} + q^{\alpha-m} \left(\begin{bmatrix} n-\beta \\ 1 \end{bmatrix} A_\beta^\perp + A_{\beta+1}^\perp \right)$ with $A_\beta^\perp \neq 0$. By the Singleton bound $d^\perp \leq \beta + 1$. Therefore, $d^\perp \leq \beta$. Since \mathcal{C} is AMRD, then $d = n - (\beta + 1)$ and we have

$$\sum_{i=0}^{n-(\beta+1)} \begin{bmatrix} n-i \\ \beta+1 \end{bmatrix} A_i = q^{\alpha-m} \sum_{j=0}^{\beta+1} \begin{bmatrix} n-j \\ \beta+1-j \end{bmatrix} A_j^\perp.$$

It follows $A_d = (q^{\alpha-m} - 1) \begin{bmatrix} n \\ \beta+1 \end{bmatrix} + q^{\alpha-m} \sum_{j=d^\perp}^{\beta} \begin{bmatrix} n-j \\ \beta+1-j \end{bmatrix} A_j^\perp + q^{\alpha-m} A_{\beta+1}^\perp$. Suppose $d^\perp < \beta$, then

$$A_d = (q^{\alpha-m} - 1) \begin{bmatrix} n \\ \beta+1 \end{bmatrix} + q^{\alpha-m} \sum_{j=d^\perp}^{\beta-1} \begin{bmatrix} n-j \\ \beta+1-j \end{bmatrix} A_j^\perp + q^{\alpha-m} \begin{bmatrix} n-\beta \\ 1 \end{bmatrix} A_\beta^\perp + q^{\alpha-m} A_{\beta+1}^\perp.$$

Then we have a contradiction since $\sum_{j=d^\perp}^{\beta-1} \begin{bmatrix} n-j \\ \beta+1-j \end{bmatrix} A_j^\perp > 0$. Hence $d^\perp = \beta$ and \mathcal{C}^\perp is AMRD.

2. Let $A_d = A_{d^\perp}^\perp$ and $\delta = t/m \in \mathbb{Z}$. Since \mathcal{C} is AMRD, then $d = n - \delta$. Therefore

$$\sum_{i=0}^{n-\delta} \begin{bmatrix} n-i \\ \delta \end{bmatrix} A_i = \sum_{j=0}^{\delta} \begin{bmatrix} n-j \\ \delta-j \end{bmatrix} A_j^\perp.$$

Then we have $A_d = \sum_{j=1}^{\delta} \begin{bmatrix} n-j \\ \delta-j \end{bmatrix} A_j^{\perp}$. By the Singleton bound $d^{\perp} \leq \delta+1$. If $d^{\perp} = \delta+1$, then $A_d = 0$, a contradiction. On the other hand, if $d^{\perp} < \delta$, then $A_d = \begin{bmatrix} n-d^{\perp} \\ \delta-d^{\perp} \end{bmatrix} A_d + \sum_{j=d^{\perp}+1}^{\delta} \begin{bmatrix} n-j \\ \delta-j \end{bmatrix} A_j^{\perp}$, a contradiction again. Hence $d^{\perp} = \delta$ and by Proposition 3.7 (5) we have \mathcal{C}^{\perp} is AMRD. \square

Example 4.5. 1. In the Example 3.5 we have $d = 1$, $d^{\perp} = 1$, $A_1 = 6$, $A_2 = 7$, $A_3 = 2$, $A_1^{\perp} = 9$, $A_2^{\perp} = 18$, $A_3^{\perp} = 4$ and $A_d = (\frac{1}{4} - 1) \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{1}{4} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} 9 + 18 \right)$, which says that \mathcal{C} is dually AMRD.

2. If $q = 3$, $m = n = 3$, $t = 3$ and

$$\mathcal{C} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle,$$

$$\text{then } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathcal{C}^{\perp}, d = 2, d^{\perp} = 1, \mathcal{C} \text{ is dually AMRD and } A_2 = A_1^{\perp} = 6.$$

Remark 4.6. It is possible that \mathcal{C} is a t -dimensional AMRD code, $t \geq m$, $m \nmid t$ and $A_d = (q^{\alpha-m} - 1) \begin{bmatrix} n \\ \beta+1 \end{bmatrix} + q^{\alpha-m} \left(\begin{bmatrix} n-\beta \\ 1 \end{bmatrix} A_{\beta}^{\perp} + A_{\beta+1}^{\perp} \right)$, but \mathcal{C} is not dually AMRD. For example, if $q = 2$, $m = n = 3$, $t = 5$ and

$$\mathcal{C} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle,$$

then $d = 1$, $d^{\perp} = 2$, $A_1 = 1$, $A_2 = 18$, $A_3 = 12$, $A_2^{\perp} = 9$, $A_3^{\perp} = 6$. Therefore \mathcal{C} is AMRD but it is not dually AMRD, even though $A_d = A_1 = (2^{-1} - 1) \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 2^{-1} (9)$. Note that this does not contradict the Theorem 4.4 because in this case $A_{\beta}^{\perp}(\mathcal{C}^{\perp}) = 0$.

Lemma 4.7. Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be a self-dual AMRD code. The following hold:

1. If $2 \nmid n$, then $d = \frac{n-1}{2}$, $m \equiv 0 \pmod{2}$ and $A_{d+1} = (q^{m/2} - \begin{bmatrix} \frac{n+1}{2} \\ 1 \end{bmatrix}) A_d - (1 - q^{m/2}) \begin{bmatrix} n \\ \frac{n+1}{2} \end{bmatrix}$.
2. If $2 \mid n$, then $d = n/2$.

Proof. We know that $2 \mid n$ if and only if $m \mid t = \frac{nm}{2}$. Therefore, if $2 \nmid n$, then $d = n - \lceil t/m \rceil = \frac{n-1}{2}$. The rest of the statement in part 1 follows from Lemma 4.3. Similarly, if $2 \mid n$, then $d = n - \lceil t/m \rceil = \frac{n}{2}$. \square

Corollary 4.8. Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be a self-dual AMRD code and $\text{char}(\mathbb{F}_q) = 2$. Then \mathcal{C} has parameters

1. $n = 3$, $t = 3\frac{m}{2}$, $d = 1$ and $4 \leq m \equiv 0 \pmod{2}$ or

2. $n = 2$, $t = m$ and $d = 1$.

Proof. By [11, Theorem 1] the all-ones matrix is in a self-dual code \mathcal{C} . Therefore $d(\mathcal{C}) = 1$. \square

Example 4.9. Let $q = 2$, $n = 2$, $m = 3$ and

$$\mathcal{C} = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Then \mathcal{C} is a 3-dimensional self-dual AMRD code.

Since in general $\lambda_{\mathcal{B}}(C^{\perp}) \neq \lambda_{\mathcal{B}}(C)^{\perp}$, if C is a self-dual AMRD code, the associated code $\lambda_{\mathcal{B}}(C) \leq (\mathbb{F}_q)_{n,m}$ is not necessarily a self-dual AMRD code. However if $\text{char}(\mathbb{F}_q) = 2$, the statement is true.

Theorem 4.10. Let $C \leq \mathbb{F}_{q^m}^n$ be a k -dimensional self-dual AMRD code. The following hold:

1. $\dim_{\mathbb{F}_q}(\lambda_{\mathcal{B}}(C)) = \dim_{\mathbb{F}_q}(\lambda_{\mathcal{B}}(C))^{\perp}$.
 2. C is a $[2d, d, d]$ code.
 3. If $n = 2$, then C is a $[2, 1, 1]$ code with $C = \langle (\alpha, 1) \rangle$, where $\alpha \in \mathbb{F}_q$ and $\alpha^2 = -1$.
 4. If $\text{char}(\mathbb{F}_q) = 2$, then $C = \langle (1, 1) \rangle$ and $\lambda_{\mathcal{B}}(C) \leq (\mathbb{F}_q)_{2,m}$ is a self-dual code.
- Proof.* 1. $\dim_{\mathbb{F}_q}(\lambda_{\mathcal{B}}(C)) = m \cdot k = \frac{nm}{2}$ and $\dim_{\mathbb{F}_q}(\lambda_{\mathcal{B}}(C)^{\perp}) = nm - \dim_{\mathbb{F}_q}(\lambda_{\mathcal{B}}(C)) = nm - mk = nm - \frac{nm}{2} = \frac{nm}{2}$.
2. Since $d = n - k = n/2$ and $k = d$, the result easily follows.
3. By part 2 we have that C is a $[2, 1, 1]$ code. Let $C = \langle (x, y) \rangle$, where $x, y \in \mathbb{F}_{q^m}$. Then $x^2 + y^2 = 0$ and $\dim_{\mathbb{F}_q} \langle x, y \rangle = 1$. Therefore there exists $\alpha \in \mathbb{F}_q$ such that $x = \alpha y$ and we have $(x, y) = y(\alpha, 1)$ with $1 + \alpha^2 = 0$.
4. If $\text{char}(\mathbb{F}_q) = 2$, then $\bar{1} := (1, \dots, 1) \in C$. In fact, if $c = (c_1, \dots, c_n) \in C$, then $0 = c \cdot c = \sum_{i=1}^n c_i^2 = (\sum_{i=1}^n c_i)^2$. Therefore $0 = \sum_{i=1}^n c_i = \bar{1} \cdot c$. Hence we have $d(C) = 1$ and $C = \langle (1, 1) \rangle$ is a $[2, 1, 1]$ code. On the other hand, if $\lambda_{\mathcal{B}}(c), \lambda_{\mathcal{B}}(c') \in \lambda_{\mathcal{B}}(C)$, then $\lambda_{\mathcal{B}}(c) = \begin{pmatrix} \lambda_{11} & \dots & \lambda_{1m} \\ \lambda_{11} & \dots & \lambda_{1m} \end{pmatrix} \in (\mathbb{F}_q)_{2,m}$ and $\lambda_{\mathcal{B}}(c') = \begin{pmatrix} \lambda'_{11} & \dots & \lambda'_{1m} \\ \lambda'_{11} & \dots & \lambda'_{1m} \end{pmatrix} \in (\mathbb{F}_q)_{2,m}$. Therefore, $\text{Tr}(\lambda_{\mathcal{B}}(c)\lambda_{\mathcal{B}}(c')^T) = 0$ and $\lambda_{\mathcal{B}}(C) \subseteq \lambda_{\mathcal{B}}(C)^{\perp}$. Since $\dim_{\mathbb{F}_q}(\lambda_{\mathcal{B}}(C)) = \dim_{\mathbb{F}_q}(\lambda_{\mathcal{B}}(C)^{\perp})$, then

$$\lambda_{\mathcal{B}}(C) = \lambda_{\mathcal{B}}(C)^{\perp}.$$

\square

Lemma 4.11. Let $C \leq (\mathbb{F}_q)_{n,m}$ be a t -dimensional dually AMRD code, with minimum distance d and $t = \frac{nm}{2} = dm$. Then C is a formally self-dual code. In particular, if $C \leq \mathbb{F}_{q^m}^n$ is a dually AMRD \mathbb{F}_{q^m} -linear $[2d, d, d]$ code with $n = 2d \leq m$, then C is formally self-dual.

Proof. Since $d = n - t/m$ and $d^\perp = t/m$, then $d = d^\perp$. By Theorem 4.4 (2) we have $A_d(\mathcal{C}) = A_{d^\perp}(\mathcal{C}^\perp) = A_d(\mathcal{C}^\perp)$. Therefore by Proposition 4.2 (2) we have $A_i(\mathcal{C}) = A_i^\perp(\mathcal{C}^\perp)$ for $i = 0, \dots, n$. \square

Lemma 4.12. *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be a t -dimensional dually AMRD code, with minimum distance d and dual minimum distance d^\perp . If $m \mid t$ and $A_{d+1} = 0$, then $t \leq \frac{nm}{2}$. In particular if $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ is a dually AMRD \mathbb{F}_{q^m} -linear $[n, k, d]$ code and $A_{d+1} = 0$, then $k \leq n/2$.*

Proof. Let $\beta := t/m \in \mathbb{Z}$. By Proposition 4.2 (2) we have $A_d \leq \frac{\begin{bmatrix} n \\ d+1 \end{bmatrix}}{\begin{bmatrix} n-d \\ 1 \end{bmatrix}}(q^m - 1)$, which is equivalent to $A_d \leq \frac{\begin{bmatrix} n \\ \beta-1 \end{bmatrix}}{\begin{bmatrix} \beta \\ 1 \end{bmatrix}}(q^m - 1)$. Similarly $A_{d^\perp} \leq \frac{\begin{bmatrix} n \\ \beta+1 \end{bmatrix}}{\begin{bmatrix} n-\beta \\ 1 \end{bmatrix}}(q^m - 1)$. Then by Proposition 4.2 (3) and Theorem 4.4 (2) we have $\frac{\begin{bmatrix} n \\ \beta+1 \end{bmatrix}}{\begin{bmatrix} n-\beta \\ 1 \end{bmatrix}} \leq \frac{\begin{bmatrix} n \\ \beta-1 \end{bmatrix}}{\begin{bmatrix} \beta \\ 1 \end{bmatrix}}$. Therefore $n - \beta \geq \beta$. \square

5 Generalized weights of dually AMRD codes

In [6] the authors define an i -MRD \mathbb{F}_{q^m} -linear code $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ as a code \mathcal{C} meeting the generalized Singleton bound i.e. $\mathcal{M}(\mathcal{C}) = n - k + i$. Additionally we say that an i -MRD \mathbb{F}_{q^m} -linear code $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ is of *degree* $\deg(\mathcal{C}) = i - 1$ if i is the minimum integer with this property. For \mathbb{F}_q -linear codes we define:

Definition 5.1. Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code of dimension t . If $a_r(\mathcal{C}) = n - \lfloor \frac{t-r}{m} \rfloor$ for an integer $r = 1 + (i-1)m \in \{1, 1+m, 1+2m, \dots, 1 + (\lceil t/m \rceil - 1)m\}$, we say that \mathcal{C} is an i -MRD \mathbb{F}_q -linear code with $r = 1 + (i-1)m$. If i is the minimum integer with this property, \mathcal{C} is called an i -MRD \mathbb{F}_q -linear code of *degree* $\deg(\mathcal{C}) = i - 1$.

One easily verifies from the definition that MRD and QMRD \mathbb{F}_q -linear codes are 1-MRD codes of degree 0. In the case of \mathbb{F}_{q^m} -linear codes, if $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ is an i -MRD \mathbb{F}_{q^m} -linear code of degree $\deg(\mathcal{C}) = i - 1$ and the dimension of its associated code $\lambda_{\mathcal{B}}(\mathcal{C})$ is t , then by Theorem 2.8 and Theorem 2.9 we have

$$n - \left\lfloor \frac{t - (1 + (i-1)m)}{m} \right\rfloor = n - k + i = \mathcal{M}_i(\mathcal{C}) = a_{im}(\lambda_{\mathcal{B}}(\mathcal{C})) = a_{1+(i-1)m}(\lambda_{\mathcal{B}}(\mathcal{C})),$$

i.e. $\lambda_{\mathcal{B}}(\mathcal{C})$ is an i -MRD \mathbb{F}_q -linear code of degree $i - 1$. The reciprocal is also true and we have that \mathcal{C} is an i -MRD \mathbb{F}_{q^m} -linear code of degree $i - 1$ if and only if $\lambda_{\mathcal{B}}(\mathcal{C})$ is an i -MRD \mathbb{F}_q -linear code of degree $i - 1$. Therefore Definition 5.1 is an appropriate generalization of the concept of i -AMR for \mathbb{F}_{q^m} -linear codes.

We know that if an \mathbb{F}_{q^m} -linear code \mathcal{C} is i -MRD, then \mathcal{C} is an $(i+1)$ -MRD \mathbb{F}_{q^m} -linear code. For \mathbb{F}_q -linear codes we have the following result.

Lemma 5.2. *If $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ is a t -dimensional i -MRD \mathbb{F}_q -linear code with $r = 1 + (i-1)m$, then \mathcal{C} is an $(i+1)$ -MRD code for $1 \leq i \leq \lceil t/m \rceil - 1$.*

Proof. We prove the statement by induction over i . One easily verifies it for $i = 1$. Assume $a_r = n - \left\lfloor \frac{t - (1 + (i-1)m)}{m} \right\rfloor$ for $r = 1 + (i-1)m$. Then we have by Theorem 2.8

$$a_r = n - \left\lfloor \frac{t - (1 + (i-1)m)}{m} \right\rfloor < a_{r+m} \leq n - \left\lfloor \frac{t - (1 + im)}{m} \right\rfloor.$$

Then

$$n - \left\lfloor \frac{t - (1 + (i-1)m)}{m} \right\rfloor < a_{r+m} \leq n - \left\lfloor \frac{t - (1 + (i-1)m)}{m} \right\rfloor + 1.$$

$$\text{Therefore } a_{1+im} = a_{r+m} = n - \left\lfloor \frac{t - (1 + (i-1)m)}{m} \right\rfloor + 1 = n - \left\lfloor \frac{t - (1 + im)}{m} \right\rfloor. \quad \square$$

Lemma 5.3. *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code of dimension t . The following facts are equivalent:*

1. $a_{1+(\lceil t/m \rceil - 1)m}(\mathcal{C}) = n$.
2. $a_{t+1-\lfloor t/m \rfloor m}(\mathcal{C}^\perp) \neq 1$.
3. *There exists $i \in \{1, \dots, \lceil t/m \rceil\}$ such that \mathcal{C} is an i -MRD code of degree $0 \leq \deg(\mathcal{C}) \leq i-1 \leq \lceil t/m \rceil - 1$.*

Proof. Interchanging \mathcal{C} with \mathcal{C}^\perp in Theorem 2.11 we have

$$\{n+1 - a_{1+t+(n-\lceil \frac{2t+1}{m} \rceil)m}^\perp, \dots, n+1 - a_{1+t-\lfloor t/m \rfloor m}^\perp\} = [n] \setminus \{a_1, a_{1+m}, \dots, a_{1+(\lceil t/m \rceil - 1)m}\}.$$

Since

$$n+1 - a_{1+t+(n-\lceil \frac{2t+1}{m} \rceil)m}^\perp < \dots < n+1 - a_{1+t-\lfloor t/m \rfloor m}^\perp,$$

then

$$\max\{n+1 - a_{1+t+(n-\lceil \frac{2t+1}{m} \rceil)m}^\perp, \dots, n+1 - a_{1+t-\lfloor t/m \rfloor m}^\perp\} = n+1 - a_{1+t-\lfloor t/m \rfloor m}^\perp.$$

Similarly $\max\{a_1, a_{1+m}, \dots, a_{1+(\lceil t/m \rceil - 1)m}\} = a_{1+(\lceil t/m \rceil - 1)m}$.

Therefore $a_{1+(\lceil t/m \rceil - 1)m} = n \Leftrightarrow n+1 - a_{1+t-\lfloor t/m \rfloor m}^\perp \neq n \Leftrightarrow a_{1+t-\lfloor t/m \rfloor m}^\perp \neq 1$.

On the other hand, if $a_{1+(\lceil t/m \rceil - 1)m} = n$, then $a_{1+(\lceil t/m \rceil - 1)m} = n - \left\lfloor \frac{t - (1 + (\lceil t/m \rceil - 1)m)}{m} \right\rfloor$, since

$$n - \left\lfloor \frac{t - (1 + (\lceil t/m \rceil - 1)m)}{m} \right\rfloor = n - \left\lfloor \frac{t-1}{m} \right\rfloor + \lceil t/m \rceil - 1 = n - \lceil t/m \rceil + 1 + \lceil t/m \rceil - 1 = n.$$

Thus \mathcal{C} is $\lceil t/m \rceil$ -MRD. Reciprocally, if \mathcal{C} is i -MRD, for $i \in \{1, \dots, \lceil t/m \rceil - 1\}$, then \mathcal{C} is $(i+j)$ -MRD for $j \geq 1$. In particular \mathcal{C} is $\lceil t/m \rceil$ -MRD code and $a_{1+(\lceil t/m \rceil - 1)m} = n$. \square

Corollary 5.4. *Let $C \leq \mathbb{F}_{q^m}^n$ be an \mathbb{F}_{q^m} -linear code. The following facts are equivalent:*

1. $\mathcal{M}_k(C) = n$.

2. $\mathcal{M}_1(C^\perp) \neq 1$.

3. There exists $i \in \{1, \dots, k\}$, such that C is an i -MRD code of degree $0 \leq \deg(C) \leq i - 1 \leq k - 1$.

Proof. In this case $t := \dim_{\mathbb{F}_q}(\lambda_{\mathcal{B}}(C)) = m \cdot k$. Then by Theorem 2.9 we have

$$\mathcal{M}_k(C) = a_{mk}(\lambda_{\mathcal{B}}(C)) = a_t(\lambda_{\mathcal{B}}(C)) = a_{1+(t/m-1)m}(\lambda_{\mathcal{B}}(C)).$$

□

Remark 5.5. 1. Note that in general if \mathcal{C} is an i -MRD \mathbb{F}_q -linear code and $m \mid t$, by Lemma 5.3 this is equivalent to $a_1(\mathcal{C}^\perp) \neq 1$, but not necessarily $a_{t/m}(\mathcal{C}) = n$.

2. When we work with the Hamming metric we have $d(C^\perp) = 1$ if and only if all vectors of C have a zero at a certain position, which is equivalent to $d_k(C) = 0$. In rank metric codes, if $C \leq \mathbb{F}_{q^m}^n$ is an \mathbb{F}_{q^m} -linear code, then $\mathcal{M}_1(C^\perp) = 1$ if and only if there exists an i -th position, such that for every vector $v = (v_1, \dots, v_n) \in C$ we have $v_i = \sum_{j \neq i} \beta_j v_j$, where $\beta_j \in \mathbb{F}_q$.

Theorem 5.6. Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code and $a_{1+(\lceil t/m \rceil - 1)m}(\mathcal{C}) = n$. Then \mathcal{C} is i -MRD with $r = 1 + (i - 1)m$ if and only if $2 + \lfloor \frac{t-r}{m} \rfloor \leq a_{t+1-\lfloor t/m \rfloor m}^\perp$. In particular, \mathcal{C} is i -MRD of degree $i - 1$ if and only if $a_{t+1-\lfloor t/m \rfloor m}(\mathcal{C}^\perp) = 2 + \lfloor \frac{t-r}{m} \rfloor$ or equivalently $a_{t+1-\lfloor t/m \rfloor m}(\mathcal{C}^\perp) = \lceil t/m \rceil - i + 2$.

Proof. Let $a_{1+(\lceil t/m \rceil - 1)m}(\mathcal{C}) = n$. We know that

$$\{n + 1 - a_{1+t+(n-\lceil \frac{2t+1}{m} \rceil)m}^\perp, \dots, n + 1 - a_{1+t-\lfloor t/m \rfloor m}^\perp\} = [n] \setminus \{a_1, a_{1+m}, \dots, a_{1+(\lceil t/m \rceil - 1)m}\}.$$

Then $n + 1 - a_{1+t-\lfloor t/m \rfloor m}(\mathcal{C}^\perp) = \max([n] \setminus \{a_1, a_{1+m}, \dots, a_{1+(\lceil t/m \rceil - 1)m}\})$. If the sequence

$$a_r(\mathcal{C}) < a_{r+m}(\mathcal{C}) < a_{r+2m}(\mathcal{C}) < \dots < a_{1+(\lceil t/m \rceil - 1)m}(\mathcal{C}) = n \quad (1)$$

with $r = 1 + (i - 1)m$ has no gaps, then $a_r(\mathcal{C}) = n - ((\lceil t/m \rceil - 1) - (i - 1))$ and $a_r(\mathcal{C}) = n - \lfloor \frac{t-r}{m} \rfloor$. Therefore by Lemma 5.2 the sequence (1) has no gaps if and only if $a_r(\mathcal{C}) = n - \lfloor \frac{t-r}{m} \rfloor$. Hence $n + 1 - a_{1+t-\lfloor t/m \rfloor m}(\mathcal{C}^\perp) \leq n - \lfloor \frac{t-r}{m} \rfloor - 1$ if and only if $a_r = n - \lfloor \frac{t-r}{m} \rfloor$.

In particular,

$$\begin{aligned} 2 + \lfloor \frac{t-r}{m} \rfloor = a_{1+t-\lfloor t/m \rfloor m}(\mathcal{C}^\perp) &\Leftrightarrow n + 1 - a_{1+t-\lfloor t/m \rfloor m}(\mathcal{C}^\perp) = n - \lfloor \frac{t-r}{m} \rfloor - 1 \\ &\Leftrightarrow a_{r-m}(\mathcal{C}) < a_r(\mathcal{C}) - 1 = n - \lfloor \frac{t-r}{m} \rfloor - 1 = n - \left\lfloor \frac{t-(r-m)}{m} \right\rfloor, \end{aligned}$$

which means that \mathcal{C} is i -MRD of degree $i - 1$. □

Corollary 5.7. Let $C \leq \mathbb{F}_{q^m}^n$ be an \mathbb{F}_{q^m} -linear code and $\mathcal{M}_k = n$. Then C is i -MRD if and only if $k - i + 2 \leq \mathcal{M}_1(C^\perp)$. In particular, C is i -MRD of degree $i - 1$ if and only if $\mathcal{M}_1(C^\perp) = k - i + 2$.

Proof. C is an i -MRD \mathbb{F}_{q^m} -linear code if and only if $\lambda_{\mathcal{B}}(C)$ is an i -MRD \mathbb{F}_q -linear code. By Theorem 5.6, this is equivalent to $\mathcal{M}_1(C^\perp) = a_{t+1-\lfloor t/m \rfloor m}^\perp(\lambda_{\mathcal{B}}(C)) \geq 2 + \lfloor \frac{t-r}{m} \rfloor = 2 + \lfloor \frac{t-(1+(i-1)m)}{m} \rfloor = k - i + 2$. Furthermore, C is an i -MRD \mathbb{F}_{q^m} -linear code of degree $i - 1$ if and only if $\lambda_{\mathcal{B}}(C)$ is also of degree $i - 1$, which by Theorem 5.6 is equivalent to $\mathcal{M}_1(C^\perp) = a_1^\perp(\lambda_{\mathcal{B}}(C)) = 2 + t/m - i = k - i + 2$. \square

Theorem 5.8. *Let $C \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code of dimension t and $m \mid t$. The following hold:*

1. *Let $a_{1+(\lfloor t/m \rfloor - 1)m}(C) = n$. Then C is an i -MRD code of degree $\deg(C) = i - 1$ if and only if C^\perp is an A^{i-1} MRD code i.e. $\text{Rdef}(C^\perp) = i - 1$.*
2. *$a_{1+(\lfloor t/m \rfloor - 1)m}(C) < n$ if and only if C is an $A^{t/m}$ AMR code i.e. $\text{Rdef}(C^\perp) = t/m$.*

Proof. 1. By Theorem 5.6, C is an i -MRD code of degree $i - 1$ if and only if $\text{Rdef}(C^\perp) = \lfloor t/m \rfloor + 1 - d^\perp = \lfloor t/m \rfloor + 1 - \lceil t/m \rceil + i - 2 = i - 1$.

2. By Lemma 5.3, we have

$$a_{1+(\lfloor t/m \rfloor - 1)m}(C) < n \Leftrightarrow a_1(C^\perp) = 1 \Leftrightarrow \text{Rdef}(C^\perp) = \lfloor t/m \rfloor + 1 - 1 = t/m.$$

\square

Corollary 5.9. *Let $C \leq \mathbb{F}_{q^m}^n$ be an \mathbb{F}_{q^m} -linear $[n, k, d]$ code. The following hold:*

1. *If $\mathcal{M}_k = n$, then C is an i -MRD code of degree $\deg(C) = i - 1$ if and only if C^\perp is an A^{i-1} MRD code i.e. $\text{Rdef}(C^\perp) = i - 1$.*
2. *$\mathcal{M}_k < n$ if and only if C is an A^k AMR code i.e. $\text{Rdef}(C^\perp) = k$.*

Theorem 5.10. *Let $C \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code of dimension t and $m \mid t$. The following facts hold:*

1. *Assume $a_{1+(\lfloor t/m \rfloor - 1)m}(C) = n$ and $\text{Rdef}(C) \geq 1$. Then C^\perp is AMRD if and only if $a_{1+m}(C) = d + \text{Rdef}(C) + 1$. Therefore, if C is AMRD, then C is dually AMRD if and only if $a_{1+m}(C) = d + 2$.*
2. *If $a_{1+(\lfloor t/m \rfloor - 1)m}(C) < n$, then C^\perp is AMRD if and only if $\lfloor t/m \rfloor = 1$. Moreover, C is dually AMRD with $t/m = 1$ if and only if $a_{1+(\lfloor t/m \rfloor - 1)m}(C) < n$ and $d = n - 1$.*

Proof. 1. Let C^\perp AMRD. Since $\text{Rdef}(C^\perp) = 1$, then By Theorem 5.8 (1) we have $\deg(C) = 1$. Therefore $a_{1+m}(C) = n - \lfloor \frac{t-(1+m)}{m} \rfloor = n - \lfloor t/m \rfloor + 2 = (n - \lfloor t/m \rfloor + 1 - d) + d + 1 = \text{Rdef}(C) + d + 1$. Reciprocally, let $a_{1+m}(C) = d + \text{Rdef}(C) + 1$. Then $a_{1+m}(C) = d + n - \lfloor t/m \rfloor + 1 - d + 1 = n - \lfloor t/m \rfloor + 2 = n - \lfloor \frac{t-(1+m)}{m} \rfloor$. Since $\text{Rdef}(C) \geq 1$, then $d \neq n - \lfloor t/m \rfloor + 1 = n - \lfloor \frac{t-1}{m} \rfloor$ and C is a 2-AMRD with $\deg(C) = 1$. Therefore $\text{Rdef}(C^\perp) = 1$.

2. The result easily follows from Theorem 5.8 (2). □

Corollary 5.11. *Let $C \leq \mathbb{F}_{q^m}^n$ be an \mathbb{F}_{q^m} -linear $[n, k, d]$ code. The following facts hold:*

1. *Assume $\mathcal{M}_k(C) = n$ and $\text{def}(C) \geq 1$. Then C^\perp is AMRD if and only if $\mathcal{M}_2(C) = d + \text{def}(C) + 1$. Therefore, if C is AMRD, then C is dually AMRD if and only if $\mathcal{M}_2(C) = d + 2$.*
2. *If $\mathcal{M}_k(C) < n$, then C^\perp is AMRD if and only if $k = 1$. Moreover, C is a 1-dimensional dually AMRD code if and only if $\mathcal{M}_k(C) < n$ and $d = n - 1$.*

Example 5.12. If $\mathcal{M}_k(C) < n$, then C is dually AMRD if and only if $k = 1$ and $\mathcal{M}_k = d = n - 1$ i.e. $C = \langle v_1, \dots, v_n \rangle$, where the maximum number of coordinates v_1, \dots, v_n that are linearly independent over \mathbb{F}_q is $n - 1$. For example, let $q = 2$, $n = m = 4$ and $\mathbb{F}_{2^4} = \mathbb{F}_2[\alpha]$, where α satisfies $\alpha^4 + \alpha + 1 = 0$. The 1-dimensional code $C \leq \mathbb{F}_{2^4}^4$ generated by $(1, \alpha, \alpha^2, 0)$ has minimum rank $d = 3$ and since $(0, 0, 0, 1) \in C^\perp$, then $d^\perp = 1$. Therefore C is dually AMRD. Another example is the code \mathcal{G} , where \mathcal{G} is the Gabidulin code with $\text{gen}(\mathcal{G}) = M_k(v_1, \dots, v_n)$ (see Theorem 3.10).

Definition 5.13. An \mathbb{F}_q -linear code $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ is called a 2-AMRD \mathbb{F}_q -linear code if and only if \mathcal{C} is AMRD and 2-MRD. Similar definition is given for an \mathbb{F}_{q^m} -linear code considering its associated code.

In coding theory under the Hamming metric, these 2-AMRD codes are called near MDS codes (see [4, 7]). We can easily see that if $m \mid t$, then the concepts dually AMRD and 2-AMRD agree, as the following theorem shows.

Theorem 5.14. *Let $\mathcal{C} \leq (\mathbb{F}_q)_{n,m}$ be an \mathbb{F}_q -linear code of dimension t with minimum distance d and dual minimum distance d^\perp . The following facts hold:*

1. *If \mathcal{C} is 2-AMRD, then $a_{1+t-(\lfloor t/m \rfloor - 1)m}^\perp(\mathcal{C}) = \lceil t/m \rceil + 2$ and $a_{1+t-\lfloor t/m \rfloor m}^\perp(\mathcal{C}) = \lceil t/m \rceil$.*
2. *If $m \mid t$ and \mathcal{C} is 2-AMRD, then \mathcal{C}^\perp is 2-AMRD.*
3. *Let $m \mid t$. Then \mathcal{C} is a 2-AMRD code if and only if \mathcal{C} is a dually AMRD code with $t/m > 1$.*

Proof. 1. Let \mathcal{C} be a 2-AMRD code. Then $a_1(\mathcal{C}) = n - \lceil t/m \rceil$, $a_{1+m}(\mathcal{C}) = n - \left\lfloor \frac{t-(1+m)}{m} \right\rfloor$ and $a_{1+m}(\mathcal{C}) = 1 + a_1(\mathcal{C})$. We see that the sequence

$$a_{1+m} < a_{1+2m} < \dots < a_{1+(\lceil t/m \rceil - 1)m} = n$$

is exactly the sequence

$$n - \left\lfloor \frac{t - (1 + m)}{m} \right\rfloor < n - \left\lfloor \frac{t - (1 + 2m)}{m} \right\rfloor < \dots < n - \left\lfloor \frac{t - (1 + (\lceil t/m \rceil - 1)m)}{m} \right\rfloor = n,$$

which has no gaps. By Theorem 2.11

$$\{n+1-a_{1+t+(n-\lceil \frac{2t+1}{m} \rceil)_m}^\perp, \dots, n+1-a_{1+t-\lfloor t/m \rfloor_m}^\perp\} = [n] \setminus \{a_1, \dots, a_{1+(\lceil t/m \rceil - 1)_m}\}.$$

Therefore $n+1-a_{1+t-\lfloor t/m \rfloor_m}^\perp(\mathcal{C}) = a_{1+m}(\mathcal{C}) - 1 = n - \lceil t/m \rceil + 2$ and

$$n+1-a_{1+t-(\lfloor t/m \rfloor - 1)_m}^\perp(\mathcal{C}) = a_1(\mathcal{C}) - 1 = n - \lceil t/m \rceil - 1.$$

Thus $a_{1+t-(\lfloor t/m \rfloor - 1)_m}^\perp(\mathcal{C}) = \lceil t/m \rceil + 2$ and $a_{1+t-\lfloor t/m \rfloor_m}^\perp(\mathcal{C}) = \lceil t/m \rceil$.

2. In this case $a_1(\mathcal{C}^\perp) = t/m$ and $a_{1+m}(\mathcal{C}) = t/m + 2$. Therefore $\text{Rdef}(\mathcal{C}^\perp) = 1$ and $a_{1+m}(\mathcal{C}^\perp) = n - \left\lfloor \frac{mn-t-(1+m)}{m} \right\rfloor$.
3. By Theorem 5.10 (1) we have that \mathcal{C} is dually AMRD with $t/m > 1$ if and only if \mathcal{C} is AMRD and $a_{1+m}(\mathcal{C}) = d + 2$, which is equivalent to \mathcal{C} being AMRD and $a_{1+m}(\mathcal{C}) = n - \lceil t/m \rceil + 2 = n - \left\lfloor \frac{t-(1+m)}{m} \right\rfloor$.

□

Corollary 5.15. *Let $C \leq \mathbb{F}_{q^m}^n$ be an \mathbb{F}_{q^m} -linear $[n, k]$ code with minimum distance d and dual minimum distance d^\perp . The following facts hold:*

1. *If C is 2-AMRD, then C^\perp is 2-AMRD.*
2. *C is a 2-AMRD code if and only if C is a dually AMRD code with dimension $k > 1$.*

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